

Complex Eigenvalues

We've dealt with real eigenvalues. In general, a matrix might have complex eigenvalues

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm \sqrt{-1} = \pm i$$

Note: A rotates by 90°

Previously, we argued geometrically that A has no real eigenvalues.

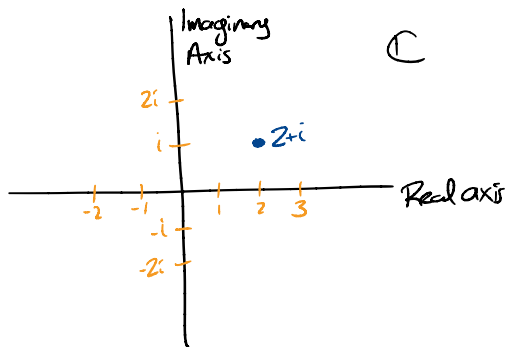
Let's review complex numbers:

A complex number is of the form $z = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ is a solution to $x^2 + 1 = 0$.

The set of complex numbers is denoted by $\mathbb{C}$

$\mathbb{C}$ can be viewed as $\mathbb{R}^2$:

Note: $\mathbb{R} \subset \mathbb{C}$



Operations:

- $(a+bi) + (c+di) = (a+c) + (b+d)i$
- $(a+bi)(c+di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)i$
- The complex conjugate of $z = a+bi$ is $\bar{z} = a-bi$.
(reflection over real axis)

The magnitude of $z = a+bi$ is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$.

↑ distance between z and 0

Properties:

$$\bullet \overline{z+w} = \bar{z} + \bar{w}$$

$$\bullet \overline{z\bar{w}} = \bar{z} w$$

(proof on Hw)

$$\bullet z = \bar{z} \text{ if and only if } z \in \mathbb{R}.$$

Let $p(x) = a_0 + a_1x + \dots + a_nx^n$, $a_0, \dots, a_n \in \mathbb{R}$.

Fact: If $z \in \mathbb{C}$ is a root of p , then so is $\bar{z}$.

proof: Let $p(z) = 0$. Then:

$$\begin{aligned} p(\bar{z}) &= a_0 + a_1\bar{z} + \dots + a_n\bar{z}^n = \overline{a_0 + a_1z + \dots + a_nz^n} \\ &= \overline{a_0 + a_1z + \dots + a_nz^n} \\ &= \overline{a_0 + a_1z + \dots + a_nz^n} = \overline{p(z)} = \bar{0} = 0 \quad \blacksquare \end{aligned}$$

Since eigenvalues are roots of the characteristic polynomial, the eigenvalues of a matrix come in conjugate pairs.

Hence every matrix has an even number of

(nonreal) complex eigenvalues

Ex: A 3×3 matrix must have a real eigenvalue

Since it has exactly 3 complex eigenvalues (counted with multiplicity) and an even number of those are complex, it has either 1 or 3 real eigenvalues.

We will briefly consider complex matrices later in the semester. For now we only consider real matrices

Symmetric Matrices

Symmetric matrices have really nice properties.

Recall, a (real) matrix S is symmetric if $S^T = S$

Let's diagonalize a symmetric matrix.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

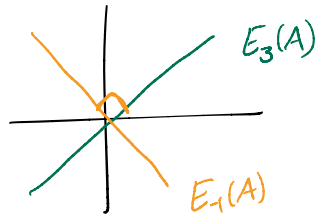
$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$\Rightarrow$ eigenvalues are $\lambda = -1, 3$ *real!*

Now we can compute the eigenspaces

$$E_{-1}(A) = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$$E_3(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \begin{matrix} \searrow \\ \rightarrow \end{matrix} \text{orthogonal!}$$



Now diagonalize using unit eigenvectors:

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \begin{matrix} \text{orthogonal} \\ \text{matrix!} \end{matrix}$$

$$\text{So } A = PDP^{-1} = PDP^T \text{ since } Q \text{ is orthogonal}$$

This happens in general!

Spectral Theorem: Let S be a (real) symmetric matrix

Then $S = Q D Q^T$ where D is diagonal
and Q is an orthogonal matrix of eigenvectors.

(S is orthogonally diagonalizable)

To prove this, we need two key ingredients.

1. the eigenvalues of S are real
2. the eigenspaces of S are orthogonal